Data consistency conditions for 2D time-of-flight PET

Two pages supporting note, Grenoble January 2017, Michel Defrise

I. Time-of-flight 2D PET data

The TOF sinogram data parameterization is

$$p(\phi, s, t) = \int_{-\infty}^{\infty} dl\, w(t-l)\, f(s\hat{u}^{\perp} + l\hat{u}) \tag{1}$$

where s and ϕ are the usual radial and angular sinogram coordinates. The unit vectors $\hat{u}^{\perp} = (\cos\phi, \sin\phi)$ and $\hat{u} = (-\sin\phi, \cos\phi)$ are orthogonal and parallel to the line of response.

The *TOF profile* w is centered at point t along the line of response. The TOF profile w is determined by the time resolution of the scanner and is usually well modeled by a gaussian with standard deviation σ[1]

$$w(t) = e^{-t^2/2\sigma^2}/\sqrt{2\pi}\sigma \leftrightarrow (\mathcal{F}w)(\nu) = \hat{w}(\nu) = e^{-2\pi^2\sigma^2\nu^2}. \tag{2}$$

The *most likely annihilation point* (MLA) corresponding to (ϕ, s, t) is defined by $\vec{r} = s\hat{u}^{\perp} + t\hat{u}$.

Inverting (1) is in essence a multi-channel deconvolution problem (one "channel" for each ϕ, see also (13) below).

II. Consistency condition for 2D TOF sinogram data

- Existence of a DCC expected from variable counting.
- Geometric interpretation on the black board.
- Trivial DCC if $\sigma \to 0$.
- The DCC (4) is *local*, unlike DCCs for non TOF 2D PET (such as Helgason-Ludwig).

If the TOF profile $w(t)$ is the gaussian profile (2) and f is smooth and decays at ∞, the 2D TOF data, equation (1), satisfy for all ϕ, s, t the partial differential equation (PDE)[2]

$$t\frac{\partial p}{\partial s} + \frac{\partial p}{\partial \phi} - s\frac{\partial p}{\partial t} + \sigma^2 \frac{\partial^2 p}{\partial s \partial t} = 0. \qquad \text{TOF-PET DCC} \tag{4}$$

The characteristic curves of (4) are helical curves of constant MLA in the (ϕ, s, t) space.

III. Equivalent DCC for 2D TOF histoimage data

The histoimage format[3] parameterizes the TOF PET data using the MLA $\vec{r}$:

$$\begin{aligned} q(\phi, \vec{r}) &= p(\phi, s, t) \text{ with } \vec{r} = s\hat{u}^{\perp} + t\hat{u} \\ &= \int_{-\infty}^{\infty} dl\, w(l)\, f(\vec{r} + l\hat{u}). \end{aligned} \tag{5}$$

This format allows a simple derivation of the DCC: if $w(t)$ is the gaussian profile (2) and f is smooth and decays at ∞, the 2D TOF data (5) satisfy for all $\vec{r}, \phi$ the DCC:

$$\frac{\partial q}{\partial \phi} + \sigma^2\, \hat{u} \cdot \nabla^2 q \cdot \hat{u}^{\perp} = 0, \quad \text{where } \hat{u} \cdot \nabla^2 q \cdot \hat{u}^{\perp} = \sum_{i,j=1}^{2} \hat{u}_i \frac{\partial^2 q}{\partial r_i \partial r_j} \hat{u}_j^{\perp}. \tag{6}$$

[1] Currently the FWHM of w is of the order of 400 ps, which corresponds to 60 mm.

[2] Proof that (4) is necessary: use $dw(t)/dt = -tw(t)/\sigma^2$, $d\hat{u}^{\perp}/d\phi = \hat{u}$ and $d\hat{u}/d\phi = -\hat{u}^{\perp}$ to rewrite the LHS as (the argument of f everywhere as in (1))

$$\begin{aligned} &\int_{-\infty}^{\infty} dl\, w(t-l) \left\{ t\hat{u}^{\perp} \cdot \nabla f + s\hat{u} \cdot \nabla f - l\hat{u}^{\perp} \cdot \nabla f + s\frac{(t-l)}{\sigma^2} f - \sigma^2 \frac{(t-l)}{\sigma^2} \hat{u}^{\perp} \cdot \nabla f \right\} \\ = &\int_{-\infty}^{\infty} dl\, w(t-l) \left\{ s\hat{u} \cdot \nabla f + s\frac{(t-l)}{\sigma^2} f \right\} = s\int_{-\infty}^{\infty} dl\, \frac{d}{dl}(w(t-l)f) = 0 \end{aligned} \tag{3}$$

Defrise M, Panin V Y, Michel C and Casey M E 2008 Continuous and discrete data rebinning in time-of-flight PET IEEE Trans. Med. Imaging 27 1310-22.

[3] Matej S, Surti S, Jayanthi S, Daube-Witherspoon M E, Lewitt R M and Karp J S 2009 Efficient 3-D TOF PET reconstruction using view-grouped histo-images: DIRECTdirect image reconstruction for TOF IEEE Trans. Med. Imaging 28 739-51

Proof.

$$
\begin{aligned}
\frac{\partial q}{\partial \phi} &= -\int_{-\infty}^{\infty} dl \;\; w(l)\, l\, (\hat{u}^{\perp} \cdot \nabla f)(\vec{r} + l\hat{u}) \text{ since } \frac{\partial \hat{u}}{\partial \phi} = -\hat{u}^{\perp} \\
\nabla^2 q &= \int_{-\infty}^{\infty} dl\, w(l)\, \nabla^2 f(\vec{r} + l\hat{u}) \qquad (7)
\end{aligned}
$$

Also, as f decays at large $\vec{x}$ and the gaussian satisfies $w'(l) = (-l/\sigma^2)w(l)$,

$$
0 = \int_{-\infty}^{\infty} dl \;\; \frac{d}{dl}\left(w(l)\, f(\vec{r} + l\hat{u})\right) = \int_{-\infty}^{\infty} dl \;\; w(l) \left(\frac{-l}{\sigma^2} f + \hat{u} \cdot \nabla f\right)(\vec{r} + l\hat{u}) \qquad (8)
$$

Taking the gradient of this equation wrt $\vec{r}$, the scalar product with $\hat{u}^{\perp}$, and multiplying by σ^2,

$$
0 = \sigma^2(\hat{u}^{\perp} \cdot \nabla)0 = \int_{-\infty}^{\infty} dl \;\; w(l) \left(-l\, \hat{u}^{\perp} \cdot \nabla f + \sigma^2\, \hat{u} \cdot \nabla^2 \cdot \hat{u}^{\perp} f\right)(\vec{r} + l\hat{u}) \qquad (9)
$$

With (7) this concludes the proof that (6) is a necessary DCC ((6) is of course equivalent to (4)).

IV. Fourier DCC and the central slice theorem for TOF PET

Consider the 2D Fourier transform

$$
\hat{q}(\phi, \vec{\nu}) = \int_{\mathbb{R}^2} d\vec{r}\, q(\phi, \vec{r})\, e^{-2\pi i\, \vec{\nu} \cdot \vec{r}}. \qquad (10)
$$

Applying the Fourier transform to (6) one obtains the necessary DCC

$$
\frac{\partial \hat{q}(\phi, \vec{\nu})}{\partial \phi} - 4\pi^2\sigma^2(\hat{u} \cdot \vec{\nu})(\hat{u}^{\perp} \cdot \vec{\nu})\, \hat{q}(\phi, \vec{\nu}) = 0 \qquad (11)
$$

which can also be written as (recall $d\hat{u}/d\phi = -\hat{u}^{\perp}$):

$$
\frac{\partial}{\partial \phi}\left(\hat{q}(\phi, \vec{\nu}) e^{2\pi^2\sigma^2(\hat{u}\cdot\vec{\nu})^2}\right) = \frac{\partial}{\partial \phi}\left(\hat{q}(\phi, \vec{\nu})/\hat{w}(\hat{u} \cdot \vec{\nu})\right) = 0. \qquad (12)
$$

But the 2D TOF projection slice theorem (Fourier transform of (5)) is[4]

$$
\begin{aligned}
\hat{q}(\phi, \vec{\nu}) &= \int_{\mathbb{R}^2} d\vec{r}\, e^{-2\pi i\, \vec{\nu}\cdot\vec{r}} \int_{-\infty}^{\infty} dl\, w(l)\, f(\vec{r} + l\hat{u}) \\
&= \int_{\mathbb{R}^2} d\vec{x}\, f(\vec{x}) \int_{-\infty}^{\infty} dl\, e^{-2\pi i\, \vec{\nu}\cdot(\vec{x} - l\hat{u})} \;\; w(l) = \hat{f}(\vec{\nu})\hat{w}(\hat{u} \cdot \vec{\nu}), \qquad (13)
\end{aligned}
$$

$\Rightarrow$ the Fourier DCC in the form (12) corresponds to the trivial identity $\partial \hat{f}(\vec{\nu})/\partial \phi = 0$. This shows that strong decay conditions on $\hat{q}$ must be required for sufficiency (i.e. $\hat{q}/\hat{w}$ must be L^2).

V. The 3D case and the link with John's equation.

The histoimage data (5) is extended to 3D with the same equation but now $\vec{r} \in \mathbb{R}^3$, $\hat{u} \in S^2$, and $\hat{u}^{\perp}$ represents the plane orthogonal to $\hat{u}$. The DCC (6) becomes a vector equation[5],

$$
\nabla_{\hat{u}} q(\hat{u}, \vec{r}) - \sigma^2\, \hat{u} \cdot \nabla^2 q(\hat{u}, \vec{r}) = \vec{0} \qquad (14)
$$

where the ∇ without subscript denotes the gradient with respect to $\vec{r}$. This equation has three components but since $\hat{u} \in S^2$, the angular gradient $\nabla_{\hat{u}}$ has a meaning only in the two directions orthogonal to $\hat{u}$. Thus 3D TOF data q must be solution of *two* independent PDEs, obtained by taking the scalar product of (14) with two unit vectors in $\hat{u}^{\perp}$.

Multiplying (14) by $(\hat{a} \cdot \nabla)(\hat{b}\cdot) - (\hat{b} \cdot \nabla)(\hat{a}\cdot)$ with $\hat{a}, \hat{b} \in \hat{u}^{\perp}$ eliminates the term in σ^2 and yields a derived DCC which also holds in the limit $\sigma \to \infty$ of non-TOF PET data: this derived equation is the John equation for the 3D x-ray transform[6].

[4] In contrast with the other equations in the note, (13) holds for any smooth TOF profile, not only for the gaussian (2).

[5] Yusheng Li et al, Phys. Med. Biol. 60 (2015) 6563-6583.

[6] F. John (1971). Partial Differential Equations. Applied Mathematical Sciences, Springer-Verlag 1971; D. V. Finch, Cone beam reconstruction with sources on a curve, SIAM J Appl Math 45, pp. 665-673, 1985; S. K. Patch, Consistency conditions upon 3D CT data and the wave equation, Phys. Med. Biol., 47, pp. 2637-2650, 2002.